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# Non-monotonic disorder-induced enhanced tunnelling

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**Abstract**
The quantum-mechanical transmission through a disordered tunnel barrier is investigated analytically in the following regime: (correlation range of the random potential) $\ll$ (penetration length) $\ll$ (barrier length). The mean and/or the width of the potential can either be constant, or vary slowly across the barrier. The typical transmission is found to be a non-monotonic function of the disorder strength, increasing at weak disorder, reaching a maximum in the crossover from weak to strong disorder, and decreasing at strong disorder. This work provides a quantitative analysis of the phenomenon of disorder-induced enhanced tunnelling, put forward by Freilikher *et al* (1995 *Phys. Rev.* E **51** 6301, 1996 *Phys. Rev.* B **53** 7413).

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## 1. Introduction

Anderson localization is one of the most spectacular disorder-induced phenomena [1]. The one-dimensional situation is especially well understood [2]. Consider for definiteness the Schrödinger equation for an electron moving on a line, in a disordered potential $V(x)$. Even in the presence of an infinitesimal amount of disorder, all eigenstates are exponentially localized, with a localization length $\xi = 1/\gamma$, where $\gamma$ is the Lyapunov exponent. As a consequence, the typical conductance of a disordered sample falls off exponentially with its length $L$. More precisely, the zero-temperature conductance $g$ of a one-channel sample is related to the transmission $T$ across the sample by the two-probe Landauer formula [3]:

$$g = \frac{2e^2}{h} T. \tag{1.1}$$

The theory of one-dimensional localization predicts that the transmission $T$ is a widely fluctuating quantity in the insulating regime ($\gamma L \gg 1$), so that the meaningful quantity

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to consider is its typical (i.e., most probable) value $\exp(\langle \ln T \rangle)$. The mean $\langle \ln T \rangle$ grows linearly with the sample length,

$$\langle \ln T \rangle \approx -2\gamma L \tag{1.2}$$

and the ratio $(\ln T)/L$ is self-averaging, in the strong sense that all the cumulants of $(\ln T)$ grow linearly with $L$ [2]. In other terms, the statistics of $T$ (resp. of $\ln T$) is similar to that of the partition function (resp. of the total free energy) of a disordered thermodynamical system. This deep analogy appears clearly in the framework of the transfer-matrix formalism, especially for discrete (tight-binding) models [4, 5].

So far, it was implicitly assumed that the energy $E$ of the incoming electron is above the mean of the disordered potential (usually taken to be zero). Much less is known on the converse situation of *tunnelling through a disordered barrier*, where the mean potential $\langle V(x) \rangle$ inside the sample is non-zero, and higher than the energy $E$. More generally, a tunnelling situation is met if the disordered sample is periodic on average, and if the energy is in a gap of the underlying average structure [6].

It has been put forward, seemingly in [6] for the first time, that a weak disorder enhances the transmission in such a tunnelling situation. In their subsequent work [7], the authors show that a weak disorder increases both the mean conductance (proportional to $\langle T \rangle$) and the mean resistance (proportional to $\langle 1/T \rangle$) of a tunnel barrier. This disorder-induced enhanced tunnelling effect is paradoxical, because random impurity potentials usually lead to additional scattering, which hinders transport. In any case, a strong enough disordered potential is expected to have the usual effect of reducing the transmission. Putting together these observations, it can therefore be anticipated that the transmission reaches a maximum in an intermediate crossover regime, corresponding to a moderate amount of disorder. This non-monotonic behaviour of the transmission as a function of the disorder strength seems to have been overlooked so far (see, however, [8] for attempts earlier than [6, 7], and [9] for a recent discussion of the effects of a weak disorder on gap states).

Our aim is to provide a quantitative analysis of the non-monotonic behaviour of the disorder-induced enhanced tunnelling transmission. We restrict this study to the regime of most physical interest:

$$a \text{ (correlation range of potential)} \ll 1/K \text{ (penetration length)} \ll L \text{ (barrier length)}. \tag{1.3}$$

The first inequality implies that the fluctuations of the disordered potential are short range, so that the latter can be modelled as a Gaussian white noise. The second inequality implies that the transmission of the barrier is exponentially small, even in the absence of disorder. This suggests that $\langle \ln T \rangle$ will be the right quantity to consider.

For completeness, we first give in section 2 an overview of tunnelling through a clean barrier. Section 3 then deals with tunnelling through a square disordered barrier (the mean and the width of the potential are constant), while section 4 is devoted to the general case (the mean and/or the width of the potential vary smoothly across the barrier). A summary and an outlook are presented in section 5.

## 2. Tunnelling through a clean barrier

We begin with a reminder of the well-known problem in quantum mechanics [10] of tunnelling through a clean barrier. In reduced units ($\hbar = 2m = 1$), the one-dimensional Schrödinger equation reads

$$-\psi''(x) + V(x)\psi(x) = E\psi(x). \tag{2.1}$$

### *2.1. Square clean barrier*

Consider first the simple case of a square barrier of length $L$. The potential is constant in the barrier:

$$V(x) = V_0 \qquad (0 \leqslant x \leqslant L) \tag{2.2}$$

and vanishes elsewhere.

In the situation of interest, the energy $E$ of the incoming particle is in the range $0 < E < V_0$. The wavevector $p$ of the particle and its inverse penetration length $K$ in the barrier read

$$p = \sqrt{E} \qquad K = \sqrt{V_0 - E}. \tag{2.3}$$

The reflection and transmission amplitudes $r$ and $t$ are determined by looking for a solution to (2.1) of the form

$$\psi(x) = \begin{cases} \mathrm{e}^{\mathrm{i}px} + r\,\mathrm{e}^{-\mathrm{i}px} & (x \leqslant 0) \\ a\,\mathrm{e}^{Kx} + b\,\mathrm{e}^{-Kx} & (0 \leqslant x \leqslant L) \\ t\,\mathrm{e}^{\mathrm{i}p(x-L)} & (x \geqslant L). \end{cases} \tag{2.4}$$

Expressing the continuity of $\psi(x)$ and of its derivative at $x = 0$ and $x = L$ provides four linear equations, whose solution yields

$$r = \frac{(p^2 + K^2)\sinh KL}{(p^2 - K^2)\sinh KL + 2\mathrm{i}pK\cosh KL} \tag{2.5}$$

$$t = \frac{2\mathrm{i}pK}{(p^2 - K^2)\sinh KL + 2\mathrm{i}pK\cosh KL}. \tag{2.6}$$

Throughout the following, we will be mostly interested in the transmission intensity coefficient (or transmission for short),

$$T = |t|^2 \tag{2.7}$$

which enters the Landauer formula (1.1). In the present case, (2.6) yields

$$T = \frac{4p^2K^2}{4p^2K^2 + (p^2 + K^2)^2\sinh^2 KL}. \tag{2.8}$$

In the regime (1.3), where the barrier length is much larger than the penetration length, the transmission falls off exponentially, as

$$T \approx \frac{16p^2K^2}{(p^2 + K^2)^2}\exp(-2KL). \tag{2.9}$$

All subsequent results for the transmission will be given *with exponential accuracy*, in analogy with (1.2). Neglecting the prefactor, we thus rewrite (2.9) as

$$\ln T \approx -2KL. \tag{2.10}$$

### *2.2. Arbitrary clean barrier*

Let us now consider tunnelling through an arbitrary clean barrier. The potential $V(x)$ is larger than the energy $E$ for $0 \leqslant x \leqslant L$, so that the inverse penetration length reads

$$K(x) = \sqrt{V(x) - E}. \tag{2.11}$$

We assume that the potential has a smooth profile across the barrier, i.e., the length scale over which $V(x)$ or $K(x)$ varies is of the order of the barrier length $L$ itself.

The transmission can be determined along the lines of the previous case, by seeking a solution to (2.1) of the form (2.4), where $\exp(\pm Kx)$ are replaced by the two elementary solutions $u(x)$ and $v(x)$, with initial values $u(0) = v'(0) = 1, u'(0) = v(0) = 0$, whose Wronskian reads $u(x)v'(x) - u'(x)v(x) = 1$. We thus obtain

$$t = \frac{2\mathrm{i}p}{p^2 v(L) + \mathrm{i}p(u(L) + v'(L)) - u'(L)}. \tag{2.12}$$

The hypothesis of a smoothly varying potential implies that it is legitimate to use the well-known WKB approximation [10, 11]. Indeed, the condition for this scheme to be valid,

$$\frac{1}{K(x)^2}\frac{\mathrm{d}K(x)}{\mathrm{d}x} \sim \frac{1}{K(x)L} \ll 1 \tag{2.13}$$

is automatically satisfied in the regime (1.3) of long barriers. Within this framework, a basis of solutions to (2.1) in the barrier reads

$$\psi_\pm(x) \sim \exp\left(\pm\int_0^x K(y)\,\mathrm{d}y\right) \qquad (0 < x < L). \tag{2.14}$$

At least one of the elementary solutions $u(x)$ and $v(x)$ (and generically both of them) is proportional to the growing solution $\psi_+(x)$. Hence (2.12) leads to the estimate

$$T \sim \frac{1}{|\psi_+(L)|^2} \tag{2.15}$$

which will be sufficient hereafter, in order to work with exponential accuracy.

The transmission therefore again falls off exponentially:

$$\ln T \approx -2\int_0^L K(x)\,\mathrm{d}x \tag{2.16}$$

where the integral is the action of the classical imaginary-time trajectory, or *instanton*, crossing the barrier at energy $E$ [10–12]. The estimate (2.16) generalizes (2.10) to a potential barrier with an arbitrary (smooth) profile.

## 3. Tunnelling through a square disordered barrier

We now turn to the case of a square disordered barrier of length $L$. The barrier potential,

$$V(x) = V_0 + W(x) \qquad (0 \leqslant x \leqslant L) \tag{3.1}$$

is the sum of a constant $V_0$ and of a disordered component $W(x)$ with zero mean.

As stated in the introduction, in regime (1.3) it is legitimate to model $W(x)$ as a Gaussian white noise, such that

$$\langle W(x)W(y)\rangle = 2D\delta(x - y). \tag{3.2}$$

Estimate (2.15) still holds in the presence of disorder. We are therefore led to consider the situation where the random potential extends over the half-line $x > 0$, and to investigate the growth rate of the generic solution to (2.1).

The most salient effect of the disordered potential is that the wavefunction in the barrier now changes sign many times in the regime under consideration, so that neither the concept of a single instanton trajectory, nor the WKB estimate (2.14), makes sense anymore. The most efficient approach to this problem is the invariant-measure method, initiated long ago by Dyson [13] and Schmidt [14]. For technical reasons, the energy $E - \mathrm{i}0$ and the inverse penetration length $K + \mathrm{i}0$ are respectively endowed with infinitesimal negative and positive imaginary parts. The method consists in introducing the Riccati variable

$$z(x) = \frac{\psi'(x)}{\psi(x)} \tag{3.3}$$

so that

$$\psi(x) = \psi(0) \exp \int_0^x z(y)\,\mathrm{d}y. \tag{3.4}$$

The Schrödinger equation (2.1) is equivalent to the Riccati equation

$$z' = K^2 - z^2 + W(x). \tag{3.5}$$

The key property of this equation is the following: the complex random variable $z(x)$ has a well-defined limit probability distribution, irrespective of the position $x$, provided it is deep enough in the sample ($Kx \gg 1$), and of the initial condition $z(0)$. Let us denote averages with respect to this invariant measure as $\langle\!\langle \cdots \rangle\!\rangle$. Equation (3.4) implies that the growing solution to (2.1) typically grows exponentially, as

$$\ln \psi_+(x) \approx \Omega x \tag{3.6}$$

where

$$\Omega = \langle\!\langle z \rangle\!\rangle \tag{3.7}$$

is the complex characteristic exponent. When the energy variable is just below the real axis, $\Omega$ splits according to [15, 5]

$$\Omega(E - \mathrm{i}0) = \gamma(E) + \mathrm{i}\pi H(E). \tag{3.8}$$

The real part of (3.8) is the Lyapunov exponent $\gamma$. Inserting the behaviour (3.6) into estimate (2.15), we therefore obtain at once the prediction

$$\langle \ln T \rangle \approx -2\gamma L \tag{3.9}$$

recalled in the introduction [see (1.2)]. The imaginary part of (3.8) is proportional to the integrated density of states of the problem per unit length,

$$H(E) = \int_{-\infty}^{E} \rho(E')\,\mathrm{d}E' \tag{3.10}$$

so that $1/H(E)$ is the mean distance between any two consecutive zeroes of $\psi_+(x)$.

The present case of a white-noise potential has been investigated by several authors [16–18]. Their findings can be recast into the following formula for the characteristic exponent:

$$\Omega = D^{1/3} F(X) \tag{3.11}$$

with

$$X = \frac{K^2}{D^{2/3}} = \frac{V_0 - E}{D^{2/3}} \tag{3.12}$$

and

$$F(X) = \mathrm{e}^{-2\mathrm{i}\pi/3} \frac{\mathrm{Ai}'(\mathrm{e}^{-2\mathrm{i}\pi/3} X)}{\mathrm{Ai}(\mathrm{e}^{-2\mathrm{i}\pi/3} X)} = \frac{\mathrm{Ai}'(X) + \mathrm{i}\,\mathrm{Bi}'(X)}{\mathrm{Ai}(X) + \mathrm{i}\,\mathrm{Bi}(X)} \tag{3.13}$$

where $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are Airy functions [19]. As a consequence of (3.8), we have

$$\gamma = D^{1/3} F^{\mathrm{R}}(X) \qquad F^{\mathrm{R}}(X) = \frac{\mathrm{Ai}(X)\mathrm{Ai}'(X) + \mathrm{Bi}(X)\mathrm{Bi}'(X)}{\mathrm{Ai}(X)^2 + \mathrm{Bi}(X)^2} \tag{3.14}$$

$$H = \frac{D^{1/3}}{\pi} F^{\mathrm{I}}(X) \qquad F^{\mathrm{I}}(X) = \frac{1}{\pi(\mathrm{Ai}(X)^2 + \mathrm{Bi}(X)^2)}. \tag{3.15}$$

References [15–18] deal separately with the real and imaginary parts of $\Omega$, and therefore rather derive (3.14) and/or (3.15). For completeness, we give in the appendix a self-contained derivation of (3.11)–(3.13).

Inserting (3.14) into (3.9) leads us to the prediction

$$\langle \ln T \rangle \approx -2D^{1/3} F^{\mathrm{R}}(X) L \tag{3.16}$$

where the scaling variable $X$ is real and positive in a tunnelling situation.

In order to emphasize the dependence of the transmission on the disorder strength, we recast the above prediction as

$$\langle \ln T \rangle \approx (\ln T)_0 G(Y) \tag{3.17}$$

where $(\ln T)_0 = -2KL$ is the result (2.10) in the absence of disorder, while

$$Y = \frac{D}{K^3} = \frac{D}{(V_0 - E)^{3/2}} = X^{-3/2} \tag{3.18}$$

is the reduced disorder strength, and the scaling function $G$ reads

$$G(Y) = Y^{1/3} F^{\mathrm{R}}(Y^{-2/3}). \tag{3.19}$$

### 3.1. Weak-disorder regime

The weak-disorder regime corresponds to $D \ll K^3$, i.e., $X \gg 1$ or $Y \ll 1$. The differential equation

$$F^2 + F' = X \tag{3.20}$$

obeyed by the function $F(X)$ easily yields the asymptotic expansion

$$F(X) = X^{1/2} - \frac{1}{4X} - \frac{5}{32X^{5/2}} + \cdots. \tag{3.21}$$

This result is formally real, so that $F^{\mathrm{R}}$ has the same asymptotic expansion (while $F^{\mathrm{I}}$ is exponentially small as $X \to +\infty$). We thus obtain

$$G(Y) = 1 - \frac{Y}{4} - \frac{5Y^2}{32} + \cdots \tag{3.22}$$

i.e., more explicitly

$$\langle \ln T \rangle \approx 2L \left( -K + \frac{D}{4K^2} + \frac{5D^2}{32K^5} + \cdots \right). \tag{3.23}$$

The first correction of order $D$ is in agreement with the perturbative result of [7]: the leading effect of a weak disorder is found to enhance transmission.

### 3.2. Strong-disorder regime

Consider now the opposite regime of strong disorder ($D \gg K^3$, i.e., $Y \gg 1$ or $X \ll 1$). In this regime, the scaling function

$$G(Y) \approx F^{\mathrm{R}}(0) Y^{1/3} \tag{3.24}$$

grows as the power $1/3$ of the strength of disorder, with the explicit prefactor

$$F^{\mathrm{R}}(0) = \frac{3^{1/3}}{2} \frac{\Gamma(2/3)}{\Gamma(1/3)} = 0.364\,506. \tag{3.25}$$

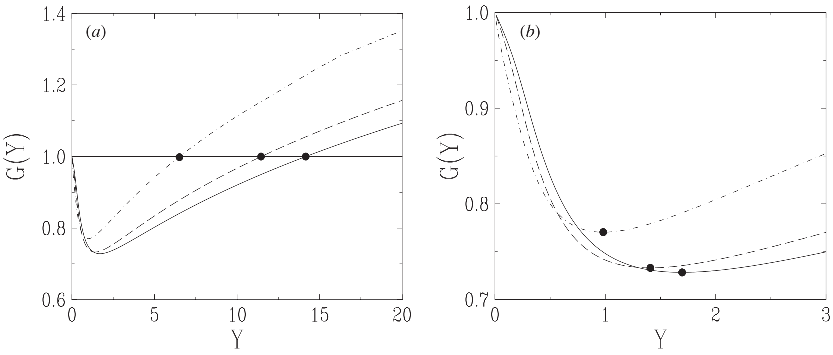


**Figure 1.** Plots of the scaling functions describing the effect of disorder on tunnelling transmission, against the reduced disorder strength $Y$. Full lines: $G$ entering (3.17) (square barrier). Dashed lines: $G_1$ entering (4.9) (parabolic barrier with parabolic disorder). Dash-dotted lines: $G_2$ entering (4.14) (parabolic barrier with uniform disorder). (*a*) Circles show the values of $Y$ where the scaling functions cross unity: $Y^{\star\star} = 14.168$, $Y_1^{\star\star} = 11.469$, $Y_2^{\star\star} = 6.522$. (*b*) (enlargement) Circles show the values of $Y$ where the scaling functions have their minima: $G = 0.7284$ for $Y^{\star} = 1.695$, $G_1 = 0.7331$ for $Y_1^{\star} = 1.408$, $G_2 = 0.7704$ for $Y_2^{\star} = 0.982$.

As a consequence, we have

$$\langle \ln T \rangle \approx -2F^{\mathrm{R}}(0)D^{1/3}L. \tag{3.26}$$

The transmission decreases with the disorder strength in the strong-disorder regime, as anticipated in the introduction. Result (3.26) is independent of $K$: the damping due to the mean barrier $V_0$ becomes negligible with respect to localization effects.

### *3.3. Non-monotonic crossover behaviour*

The function $G(Y)$ which enters the scaling law (3.17) decreases first from its value $Y(0) = 1$, according to (3.22), in the weak-disorder regime, and then increases according to (3.24), in the strong-disorder regime. It must therefore reach a minimum, somewhere in the crossover from weak to strong disorder.

Figure 1 shows plots (full lines) of the scaling function $G(Y)$, given by (3.14), (3.19). This function passes through a minimum, $G(Y^{\star}) = 0.7284$ for $Y^{\star} = 1.695$, before it crosses the value $G(Y^{\star\star}) = 1$ for $Y^{\star\star} = 14.168$. The non-monotonic behaviour of the transmission as a function of the disorder strength in the crossover regime, anticipated in the introduction, is therefore confirmed at a quantitative level. The tunnelling transmission is enhanced for a weak enough reduced disorder strength ($Y < Y^{\star\star}$), the enhancement being maximal for $Y = Y^{\star}$.

## 4. Tunnelling through an arbitrary disordered barrier

We finally turn to the general situation where both the deterministic part $V_0(x)$ and the strength of the disordered part $W(x)$ of the potential may have a smooth dependence on the position across the barrier. We set

$$V(x) = V_0(x) + W(x) \qquad (0 < x < L) \tag{4.1}$$

with

$$K(x) = \sqrt{V_0(x) - E} \tag{4.2}$$

and

$$\langle W(x)W(y)\rangle = 2D(x)\delta(x-y). \tag{4.3}$$

We assume that $K(x)$ and $D(x)$ vary smoothly across the barrier, and we again focus our attention onto the regime (1.3). In this situation, the Riccati variable $z(x)$ is approximately distributed according to the local invariant measure, characterized by the parameters $K(x)$ and $D(x)$. The reason why this adiabatic approach is legitimate is the same as for the WKB scheme in the absence of disorder: the length over which parameters vary, i.e., the barrier length $L$ itself, is much larger than the characteristic length of the relaxation of the distribution of the Riccati variable, i.e., the localization length $1/\gamma(x)$.

Equations (2.15), (3.4) yield the following expression:

$$\langle \ln T\rangle \approx -2\int_0^L \gamma(x)\,\mathrm{d}x \tag{4.4}$$

for the mean logarithm of the transmission. Prediction (4.4) includes all the previous results (2.10), (2.16), (3.9) as special cases. By means of (3.14), it can be recast into the more explicit form

$$\langle \ln T\rangle \approx -2\int_0^L D(x)^{1/3} F^{\mathrm{R}}\left(\frac{K(x)^2}{D(x)^{2/3}}\right)\mathrm{d}x. \tag{4.5}$$

In the weak-disorder regime, this prediction behaves as

$$\langle \ln T\rangle \approx 2\int_0^L \left(-K(x) + \frac{D(x)}{4K(x)^2} + \cdots\right)\mathrm{d}x \tag{4.6}$$

while in the strong-disorder regime we have

$$\langle \ln T\rangle \approx -2F^{\mathrm{R}}(0)\int_0^L D(x)^{1/3}\,\mathrm{d}x. \tag{4.7}$$

These two expressions respectively generalize (3.23) and (3.26).

To close up, let us illustrate more explicitly the result (4.5) on two examples.

**Example 1: Parabolic barrier with parabolic disorder**

In our first example, both the deterministic potential and the disorder strength have a parabolic shape:

$$K(x)^2 = V_0(x) - E = K_0^2\frac{4x(L-x)}{L^2} \qquad D(x) = D_0\frac{4x(L-x)}{L^2} \tag{4.8}$$

with maximum values $K_0^2$ and $D_0$ at the centre of the barrier ($x = L/2$).

In this situation, (4.5) yields the scaling law

$$\langle \ln T\rangle \approx (\ln T)_0 G_1(Y) \tag{4.9}$$

similar to (3.17), with

$$(\ln T)_0 = -\frac{\pi}{2}K_0 L \tag{4.10}$$

$$Y = \frac{D_0}{K_0^3} \tag{4.11}$$

and

$$G_1(Y) = \frac{2}{\pi}Y^{1/3}\int_0^{\pi} F^{\mathrm{R}}(Y^{-2/3}(\sin\theta)^{2/3})(\sin\theta)^{5/3}\,\mathrm{d}\theta. \tag{4.12}$$

The latter expression is obtained by setting $x = L(1+\cos\theta)/2$, so that $4x(L-x)/L^2 = \sin^2\theta$, with $0 \leqslant \theta \leqslant \pi$.

The scaling function $G_1(Y)$ has been evaluated numerically by means of the integral (4.12), and plotted in figure 1 (dashed lines). Its qualitative dependence on the disorder strength $Y$ is similar to that of $G(Y)$, with the following behaviour at weak and strong disorder:

$$\begin{aligned} G_1(Y) &\approx 1 - \frac{Y}{\pi} && (Y \ll 1) \\ G_1(Y) &\approx \frac{12^{1/3}\Gamma(1/3)}{5\pi} Y^{1/3} \approx 0.390\,454 Y^{1/3} && (Y \gg 1) \end{aligned} \tag{4.13}$$

with a minimum $G_1(Y_1^\star) = 0.7331$ for $Y_1^\star = 1.408$ and with $G_1(Y_1^{\star\star}) = 1$ for $Y_1^{\star\star} = 11.469$.

**Example 2: Parabolic barrier with uniform disorder**

In our second example, the deterministic potential still has the parabolic form (4.8), while the disorder strength $D(x) = D$ is constant.

Prediction (4.5) now yields the scaling form

$$\langle \ln T \rangle \approx (\ln T)_0 G_2(Y) \tag{4.14}$$

again with (4.10), and with

$$Y = \frac{D}{K_0^3} \tag{4.15}$$

and

$$G_2(Y) = \frac{2}{\pi} Y^{1/3} \int_0^\pi F^{\mathrm{R}}(Y^{-2/3}\sin^2\theta)\sin\theta\,\mathrm{d}\theta. \tag{4.16}$$

The scaling function $G_2(Y)$ is also plotted in figure 1 (dash-dotted lines). Its qualitative dependence on the disorder strength $Y$ is again similar to that of $G(Y)$, with the following behaviour at weak and strong disorder:

$$\begin{aligned} G_2(Y) &\approx 1 - \frac{Y}{3\pi}\ln\frac{1}{Y} && (Y \ll 1) \\ G_2(Y) &\approx \frac{2\,3^{1/3}}{\pi}\frac{\Gamma(2/3)}{\Gamma(1/3)} Y^{1/3} \approx 0.464\,103 Y^{1/3} && (Y \gg 1) \end{aligned} \tag{4.17}$$

with a minimum $G_2(Y_2^\star) = 0.7704$ for $Y_2^\star = 0.982$ and with $G_2(Y_2^{\star\star}) = 1$ for $Y_2^{\star\star} = 6.522$.

## 5. Discussion

In this paper, we have investigated by analytical means the transmission through a disordered tunnel barrier, in the regime (1.3) of most physical interest. We have thus provided a quantitative analysis of the phenomenon of disorder-induced enhanced tunnelling, put forward by Freilikher *et al* [6, 7]. The most salient outcome of the present work is that the enhancement effect is a non-monotonic function of the strength of disorder, and that it is maximally efficient at some well-defined intermediate value $Y^\star$ of the reduced disorder strength $Y$.

The key point of our approach consists in utilizing the scaling law (3.11)–(3.13) for the complex characteristic exponent $\Omega$. It is worth noting that this formalism encompasses, and treats on the same footing, both the usual situation of Anderson localization, where the energy is above the mean of the disordered potential (corresponding to negative values of the scaling variable $X$), and the tunnelling situation, where the energy is below the mean of the disordered potential (corresponding to positive values of $X$). As a consequence, the general results on the statistics of the transmission, recalled in the introduction, still hold in the tunnelling situation. In particular, $\langle \ln T \rangle$ is the right quantity to consider.

In the case of a white-noise potential, considered throughout this work, form (3.12) of the scaling variable $X$ is merely dictated by dimensional analysis, while the explicit formula (3.13) for the scaling function $F(X)$ is given a self-contained derivation in the appendix. Somewhat equivalent results had been obtained in several earlier works [16–18]. The scaling law (3.11)–(3.13) also describes the spectra of other one-dimensional disordered systems, such as the diffusion of classical particles in a random force field [20]. An analogous scaling formula holds for discrete models near their band edges. Consider the tight-binding equation

$$\psi_{n-1} + \psi_{n+1} + V_n \psi_n = \mathcal{E} \psi_n. \tag{5.1}$$

For a clean chain ($V_n = 0$), the dispersion relation reads $\mathcal{E} = 2 \cos p$. In the regime of a weak disorder $\left(\langle V_n^2 \rangle = \Delta \ll 1\right)$, and near the upper band edge ($p \ll 1$), one has [21]

$$\Omega = \Delta^{1/3} f(x) \qquad x = -\frac{p^2}{\Delta^{2/3}} \approx \frac{\mathcal{E} - 2}{\Delta^{2/3}}. \tag{5.2}$$

This scaling law turns out to play a central role in various problems, such as the spreading dynamics of a wave packet [22]. The results (3.11)–(3.13) can be viewed as the formal continuum limit of (5.2). The identification between our continuum problem and the tight-binding model (5.1) has to take place near the band edge. Indeed, introducing explicitly the lattice spacing $a$, which plays the role of the correlation range of the potential, the first inequality of (1.3) implies $|pa| = Ka \ll 1$, as $p = \mathrm{i}K$, whereas $D = \Delta/(2a^3)$. The scaling variables and functions match between (5.2) and (3.11)–(3.13) (up to powers of 2 due to different conventions): $x = 2^{2/3} X$, $f = 2^{1/3} F$.

For a square disordered barrier, the main result (3.17) for $\langle \ln T \rangle$, i.e., the logarithm of the typical transmission, can be generalized to all the moments of $T$ in the regime (1.3). It can indeed be shown, along the lines of [2], that these moments scale as

$$\langle T^n \rangle \approx \exp(-2n G_n(Y) K L) \tag{5.3}$$

where $Y$ is the reduced disorder strength of (3.18), while the scaling function $G_n(Y)$ depends on the order $n$ (not necessarily an integer). Equation (3.17) is recovered as $G_0(Y) = G(Y)$. Skipping every detail, let us mention the weak-disorder expansion

$$G_n(Y) = 1 - \frac{2n+1}{4} Y + \cdots \qquad (Y \ll 1) \tag{5.4}$$

hence

$$\langle T^n \rangle \approx \exp\left[2L\left(-nK + \frac{n(2n+1)D}{4K^2} + \cdots\right)\right]. \tag{5.5}$$

In agreement with [7], both $\langle T \rangle$ and $\langle 1/T \rangle$, and indeed all the moments whose order is not in the range $-1/2 < n < 0$, are increased by a weak amount of disorder. In the opposite strong-disorder regime, we have

$$G_n(Y) \approx a_n Y^{1/3} \qquad (Y \gg 1) \tag{5.6}$$

with some $n$-dependent amplitude $a_n$, hence

$$\langle T^n \rangle \approx \exp(-2n a_n D^{1/3} L). \tag{5.7}$$

The evaluation of the full scaling function $G_n(Y)$ is a difficult task in general, except for $n$ a negative integer, corresponding by means of (2.15) to positive integer powers of $|\psi_+(L)|^2$. In this case, $G_n(Y)$ can be derived by means of the algebraic approach of [2]: it is (the real part of) an algebraic function with degree $2|n| + 1$.

For a disordered barrier with an arbitrary profile, the general prediction (4.5) has been illustrated on two realistic examples of parabolic barriers. These examples demonstrate that the qualitative features of disorder-induced enhanced tunnelling, and chiefly its non-monotonic behaviour as a function of the disorder strength, are rather insensitive to the shape of the mean and/or the width of the random potential. Even quantitative characteristics, such as $Y^\star$ or $Y^{\star\star}$, do not depend too much on the profile of the barrier.

Besides the present situation of thick tunnel barriers, the non-monotonic enhancement of transmission may be a more general phenomenon. Somewhat similar features have indeed been observed recently [23] in a problem inspired by nuclear fission.

It is also worth noting that, in the somewhat dual case of the total reflection of an electron by a semi-infinite disordered sample, universal features of the distribution of the Wigner time delay have received much attention recently [24]. The effects of disorder-induced enhanced tunnelling on the Wigner time delay are also potentially of interest in transmission [25], even though this concept has been criticized as being somewhat ambiguous [26].

Finally, the disorder-induced enhancement of transmission is a quantum-mechanical phenomenon, and more generally a wave phenomenon. It is indeed due to the existence of non-trivial localized states in a weakly-disordered barrier, which themselves originate in the interferences between the multiply scattered waves. Therefore, no disorder-induced enhancement is to be expected, for example, in the Kramers problem [27] of the thermally activated hopping of a classical particle over a potential barrier.

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## Appendix. Derivation of expressions (3.11)–(3.13)

This appendix presents a self-contained derivation of expressions (3.11)–(3.13) for the complex characteristic exponent $\Omega$, introduced in (3.6).

To do so, we determine the invariant measure of the complex Riccati variable $z$, introduced in (3.3). As this distribution has a complex support, instead of writing a Fokker–Planck equation for its density, it is advantageous to consider the linear transforms

$$\Phi(y) = \langle\langle \ln(y-z) \rangle\rangle \qquad \phi(y) = \Phi'(y) = \left\langle\left\langle \frac{1}{y-z} \right\rangle\right\rangle. \tag{A.1}$$

For definiteness, we assume that $K$ has positive real and imaginary parts. Equation (3.5) shows that $z(x)$ keeps a positive imaginary part, so that $\Phi(y)$ is analytic in the lower half plane.

In the spirit of the derivation of the Fokker–Planck equation [28], we consider a small increment $\varepsilon$ of $x$, and set $z(x+\varepsilon) = z(x) + \eta$. Both

$$\langle \eta \rangle \approx (K^2 - z^2)\varepsilon \qquad \langle \eta^2 \rangle \approx 2D\varepsilon \tag{A.2}$$

are proportional to the increment $\varepsilon$, while higher cumulants are negligible.

Along the lines of the Dyson–Schmidt approach [13, 14], we then look for a stationarity condition by comparing the expressions of $\Phi(y)$ corresponding to the points $x$ and $x+\varepsilon$: $\langle\langle \ln(y-z) \rangle\rangle = \langle\langle \ln(y-z-\eta) \rangle\rangle$. By expanding the right-hand side of this equality in powers of $\eta$, and using (A.2), we obtain the condition

$$\left\langle\left\langle \frac{K^2 - z^2}{y-z} + \frac{D}{(y-z)^2} \right\rangle\right\rangle = 0 \tag{A.3}$$

i.e.,

$$D\phi'(y) + (y^2 - K^2)\phi(y) = y + \Omega. \tag{A.4}$$

The strength of disorder $D$ can be scaled out by setting $K^2 = D^{2/3}X$, $\Omega = D^{1/3}F$, proving thus the scaling formulae (3.11), (3.12) and $y = D^{1/3}u$, $\phi = D^{-1/3}\psi$.

We are thus left with a differential equation for $\psi(u)$,

$$\psi'(u) + (u^2 - X)\psi(u) = u + F \tag{A.5}$$

which can be solved by *varying the constant*:

$$\psi(u) = \exp(-u^3/3 + Xu)C(u) \qquad C(u) = \int (v + F)\exp(v^3/3 - Xv)\,\mathrm{d}v. \tag{A.6}$$

The existence of a regular solution, such that $\psi(u) \to 0$ as $|u| \to \infty$, determines $F(X)$. We must have $C(u) \to 0$ as $|u| \to \infty$ in all the directions of the lower half plane where $\exp(-u^3/3)$ diverges. This happens in two Stokes sectors, represented by the directions $u \to -\infty$ and $u \to \mathrm{e}^{-\mathrm{i}\pi/3}\infty$. We thus obtain

$$F = \frac{I'(X)}{I(X)} \tag{A.7}$$

with

$$I(X) = \int_{-\infty}^{\mathrm{e}^{-\mathrm{i}\pi/3}\infty} \exp(v^3/3 - Xv)\,\mathrm{d}v. \tag{A.8}$$

Finally, $I(X)$ is equal to the Airy function $\mathrm{Ai}(\mathrm{e}^{-2\mathrm{i}\pi/3}X)$ [19], up to a multiplicative constant. This observation leads to (3.13).

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